Lecture 3

20 February 2020 20:02

LIE ALGEBROIDS

(references: Sections 2.2 and 2.3.2 of my thesis)

Definition Proposition Exercise

(I) VECTOR BUNDLE PRELIMINARIES

Standard vector bundle constructions: direct sum, tensor product, dual on Vect_M

Pull-back bundle $\begin{array}{c} E \\ {\scriptstyle\varepsilon}\downarrow \\ M \end{array}$ $\quad \phi: N \to M \quad \phi^* E := N \,{}_{\phi}\!\times_{\varepsilon} E$

Vector bundle product $\begin{array}{cc} E_1 & E_2 \\ \downarrow & \downarrow \\ M_1 & M_2 \end{array}$ $\quad \begin{array}{c} E_1 \boxplus E_2 := pr_1^* E_1 \oplus pr_2^* E_2 \\ \downarrow \\ M_1 \times M_2 \end{array}$

Sections $\Gamma(E) := \{ s: M \to E \mid \varepsilon \circ s = id_M \}$

Note the functor $C^\infty: \mathrm{Man} \to \mathrm{Ring}$, the assignment $\Gamma: \mathrm{Vect}_{\mathrm{Man}} \to \mathrm{RMod}$ fails to be a functor since it is not possible to define module morphisms from bundle morphisms. Let us explore this in some detail:

a general bundle morphism gives the diagram:

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d] & \varphi^* E_2 \arrow[r, "id_{E_2}"] \arrow[d] & E_2 \arrow[d] \\ M_1 \arrow[r, "id_M"'] & M_1 \arrow[r, "\varphi"'] & M_2 \end{tikzcd}$$

Note that duality of bundle morphisms is not well-defined in general. At the level of sections this diagram becomes:

$$\Gamma(E_1) \xrightarrow{F} \Gamma(\varphi^* E_2) \xleftarrow{\varphi^*} \Gamma(E_1)$$

$$s_1 \mapsto F \circ s_1 \,,\, s_2 \circ \varphi \leftarrow\!\shortmid s_2$$

F-relatedness

$$s_1 \sim_F s_2 \iff F \circ s_1 = s_2 \circ \varphi$$

When $\varphi: M_1 \to M_2$ is a diffeomorphism the push-forward is defined

$$F_*: \Gamma(E_1) \to \Gamma(E_2) \qquad F_*(s+r) = F_* s + F_* r$$

$$s \mapsto F \circ s \circ \varphi^{-1} \qquad F_*(f \cdot s) = (\varphi^{-1})^* f \cdot F_* s$$

Pull-backs at the level of dual sections are always defined:

$$F^*: \Gamma(E_2^*) \to \Gamma(E_1^*)$$
$$\alpha \mapsto F^*\alpha$$

$F^*\alpha|_x(e) := \alpha_{\varphi(x)}(F(e))$

naturally extends to all tensor powers of the dual bundles

$\forall\, \eta, \omega \in \Gamma(\otimes^\bullet E_2^*)$, $f \in \Gamma(\otimes^0 E_2^*) = C^\infty(M_2)$

$$F^*(\eta+\omega) = F^*\eta + F^*\omega, \quad F^*(\omega\otimes\eta) = F^*\omega\otimes F^*\eta, \quad F^*f = \varphi^* f$$

This motivates the definition of the vector bundle analogue of the C^∞ functor:

tensor functor

$$\mathcal{T}: \mathrm{Vect}_{\mathrm{Man}} \to \mathrm{AssAlg}$$
$$E \mapsto \Gamma(\otimes^\bullet E^*) = \bigoplus_{k=0}^{\infty} \Gamma(E^*\otimes \overset{k}{\cdots} \otimes E^*)$$
$$F \mapsto F^*$$

Proposition 3.1 $\mathcal{T}$ is a functor

Proof.

This functor can be regarded as a restriction of the C^∞ functor for fibre-wise polynomial functions on the total spaces of vector bundles.

These functions are generated multiplicatively by spanning functions:

$$C^\infty_s(E) := \varepsilon^* C^\infty(M) \oplus \ell\,\Gamma(E^*)$$

Proposition 3.2 $C^\infty_s(E) \subset C^\infty(E)$ is $C^\infty(M)$-submodule and $\mathrm{span}_{\mathbb{R}}(d\,C^\infty_s(E)) = T^*E$

Proof: (Proposition 2.2.1 in my thesis)

II DEFINITION AND EXAMPLES OF LIE ALGEBROIDS

Lie algebroid $\left(\begin{smallmatrix}A\\ \downarrow\\ M\end{smallmatrix}, \rho: A\to TM, [\,,\,]\right)$ s.t. $[a, f\cdot b] = \rho_* a[f]\cdot b + f\cdot [a,b]$

Proposition 3.3 $\rho_*: \Gamma(A) \to \Gamma(TM)$ is a Lie algebra morphism

Proof: Follows as a direct corollary of Proposition 4.2 Symbol-Squiggle theorem.

Isotropy Lie algebras $(\mathfrak{g}_x, [\,,\,]_x) := (\mathrm{Ker}\,\rho_x, [\,,\,]|_{\Gamma(A)|_x})$

Characteristic distribution $\rho(A) \subset TM$ involutive by Proposition 3.3

Some natural examples

- $\begin{smallmatrix}\mathfrak{g}\\ \downarrow\\ \{\cdot\}\end{smallmatrix}$ Lie algebras
- $\begin{smallmatrix}TM\\ \downarrow\\ M\end{smallmatrix}$ tangent bundles
- $D \overset{i}{\hookrightarrow} TM$ involutive distributions
- $\begin{smallmatrix}\mathbb{R}_M\\ \downarrow\\ M\end{smallmatrix}$ vector fields $X \in \Gamma(TM)$ $\rho_x: a \mapsto a\cdot X_x$ $[f,g]_X = fX[g] - gX[f]$
- Atiyah algebroids $0 \to P\times_G \mathfrak{g} \to A_P \to TM \to 0$
- Bundles of derivations $\begin{smallmatrix}DE\\ \downarrow\\ M\end{smallmatrix}$ (see Lecture 4)

III ALGEBRAIC STRUCTURES ASSOCIATED WITH LIE ALGEBROIDS

Gerstenhaber algebra $(\Gamma(\Lambda^\bullet A), \wedge, [\![\,,]\!])$ graded commutative Lie algebra s.t. $\begin{cases} [\![a,b]\!] = [a,b] \\ [\![a,f]\!] = \rho_* a[f] \\ [\![f,g]\!] = 0 \end{cases}$

exterior algebra / de Rham complex / Lie algebroid cohomology (with trivial coefficients)

(dual construction) $(\Gamma(\Lambda^\bullet A^*), \wedge, d_A)$ differential graded algebra, $\Omega^\bullet(A)$, $H^\bullet(A)$

Cartan calculus (with trivial coefficients)

$\mathcal{L}_a = i_a d_A + d_A i_a$, $[\mathcal{L}_a, \mathcal{L}_b] = \mathcal{L}_{[a,b]}$, $[\mathcal{L}_a, i_b] = i_{[a,b]}$

Linear Poisson structure $(C^\infty(E), \{\,,\})$ s.t. $\{\ell\Gamma(E^*), \ell\Gamma(E^*)\} \subset \ell\Gamma(E^*)$, $\{\varepsilon^* C^\infty(M), \ell\Gamma(E^*)\} \subset \varepsilon^* C^\infty(M)$

Proposition 3.4 There is a 1:1 correspondence between Lie algebroids and linear Poisson structures

Proof: (Proposition 2.4.1 in my thesis) sketch: use injectivity of ℓ and α^* and define
$$\{\ell_a, \ell_b\} = \ell_{[a,b]}$$
$$\{\ell_a, \alpha^* f\} = \alpha^* \rho_* a[f]$$

IV MORPHISMS OF LIE ALGEBROIDS

Since Lie algebroids have been defined as vector bundles, it is desirable that a morphism of Lie algebroids is a vector bundle morphism satisfying some extra compatibility conditions with the Lie algebroid structure.

Let us consider the general situation of a vector bundle morphism between Lie algebroids

$$F : (A_1, \rho_1, [\,,]_1) \to (A_2, \rho_2, [\,,]_2)$$
$$\varphi : M_1 \longrightarrow M_2$$

Demanding **compatibility with the anchor** is natural from the point of view of morphisms between vector bundles:

$$\rho_2 \circ F = T\varphi \circ \rho_1$$

However, we saw in I that push-forwards are only defined for φ diffeomorphism. At the level of sections, the most we can demand is **compatibility with the brackets**

$$\left.\begin{matrix} a_1 \sim_F a_2 \\ b_1 \sim_F b_2 \end{matrix}\right\} \Rightarrow [a_1, b_1]_1 \sim_F [a_2, b_2]_2$$

Not all sections of the image $\Gamma(F(A_1))$ are necessarily F-related to some section $\Gamma(A_1)$ and thus it is desirable to find a requirement that is more comprehensive and, ideally, explicitly in terms of morphisms of the appropriate algebraic structures.

From III we see how different characterisations of Lie algebroid structures may help defining the general notion of morphism:

i) Firstly, if we insist in having morphisms of vector bundles, we should look at the DGA characterisation of Lie algebroids and notice that pull-backs restrict to exterior algebra morphisms:

$$F^* : (\Gamma(\Lambda^\bullet A_2), \wedge) \to (\Gamma(\Lambda^\bullet A_1), \wedge)$$

Then, we say that F is a morphism of Lie algebroids if F^* is a morphism of the associated DGAs, that is,

$$F^* \circ d_{A_2} = d_{A_1} \circ F^*$$

It follows from the explicit definition of d_A that a morphism of Lie algebroids in this sense is necessarily compatible with the anchor and the brackets.

ii) Since Lie algebroids are in 1:1 correspondence with linear Poisson structures, we may restrict to the appropriate notion of coisotropic relation to define a more general notion of Lie algebroid morphism.